COMP4161: Advanced Topics in Software Verification

$\lambda^{\rightarrow}$

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S2/2017

data61.csiro.au

Last time...

- λ calculus syntax
- free variables, substitution
- β reduction
- α and η conversion
- β reduction is confluent
- λ calculus is expressive (turing complete)
- λ calculus is inconsistent (as a logic)

Content

- ➜ Intro & motivation, getting started [1]
- ➜ Foundations & Principles
  - Lambda Calculus, natural deduction [1,2]
  - Higher Order Logic [3[a]]
  - Term rewriting [4]
- ➜ Proof & Specification Techniques
  - Inductively defined sets, rule induction [5]
  - Datatypes, recursion, induction [6, 7]
  - Hoare logic, proofs about programs, C verification [8[b],9]
  - (mid-semester break)
  - Writing Automated Proof Methods [10]
  - Isar, codegen, typeclasses, locales [11[c],12]

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[a] a1 due; [b] a2 due; [c] a3 due

λ calculus is inconsistent

Can find term R such that $R\ R =_\beta \texttt{not}(R\ R)$

There are more terms that do not make sense:

$$1\ 2, \quad \texttt{true}\ \texttt{false}, \quad \text{etc.}$$

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Solution: rule out ill-formed terms by using types.
(Church 1940)

Introducing types

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- if x has type α then $\lambda x.\ x$ is a function from α to α
  Write: $(\lambda x.\ x) :: \alpha \Rightarrow \alpha$
- for $s\ t$ to be sensible:
  s must be a function
  t must be right type for parameter

  If $s :: \alpha \Rightarrow \beta$ and $t :: \alpha$ then $(s\ t) :: \beta$

That's about it

Now formally again

Syntax for $\lambda^{\rightarrow}$

Terms: $t ::= v \mid c \mid (t\ t) \mid (\lambda x.\ t)$

$v, x \in V, \quad c \in C, \quad V, C$ sets of names

Types: $\tau ::= \mathtt{b} \mid \nu \mid \tau \Rightarrow \tau$

$\mathtt{b} \in \{\mathtt{bool}, \mathtt{int}, \ldots\}$ base types

$\nu \in \{\alpha, \beta, \ldots\}$ type variables

$$\alpha \Rightarrow \beta \Rightarrow \gamma \quad = \quad \alpha \Rightarrow (\beta \Rightarrow \gamma)$$

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A term t is **well typed** or **type correct**
if there are Γ and τ such that $\Gamma \vdash t :: \tau$

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Variables: $$\frac{}{\Gamma \vdash x :: \Gamma(x)}$$

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Formally:

$$\Gamma \vdash t :: \tau \quad \Longrightarrow \quad \exists \sigma.\ \Gamma \vdash t :: \sigma \wedge (\forall \sigma'.\ \Gamma \vdash t :: \sigma' \Longrightarrow \sigma' \lesssim \sigma)$$

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Type checking and type inference on $\lambda^{\rightarrow}$ are decidable.

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This property is called **subject reduction**

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- **$=_\beta$ is decidable**
  To decide if $s =_\beta t$, reduce s and t to normal form (always exists, because $\longrightarrow_\beta$ terminates), and compare result.
- **$=_{\alpha\beta\eta}$ is decidable**
  This is why Isabelle can automatically reduce each term to $\beta\eta$ normal form.

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Each computable function can be encoded as closed, type correct $\lambda^{\rightarrow}$ term using $Y :: (\tau \Rightarrow \tau) \Rightarrow \tau$ with $Y\ t \longrightarrow_{\beta} t\ (Y\ t)$ as only constant.

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- ➜ Y is called fix point operator
- ➜ used for recursion
- ➜ lose decidability (what does $Y\ (\lambda x.\ x)$ reduce to?)
- ➜ (Isabelle/HOL doesn't have Y; it supports more restricted forms of recursion)

Types and Terms in Isabelle

Types: $\tau \quad ::= \quad \mathtt{b} \mid {'\nu} \mid {'\nu} :: C \mid \tau \Rightarrow \tau \mid (\tau, \ldots, \tau)\ K$

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  Example: $\alpha :: \mathit{order}$
- **schematic variables**: variables that can be instantiated.

Type Classes

➜ similar to Haskell's type classes, but with semantic properties

class order =

assumes order_refl: "$x \leq x$"

assumes order_trans: "$[\![x \leq y; y \leq z]\!] \Longrightarrow x \leq z$"

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- ➜ can be instantiated

  instance nat :: "{order, linorder}" **by** ...

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Solution:
Isabelle has **free** (x), **bound** (x), and **schematic** (?X) variables.

Only schematic variables can be instantiated.

Free converted into schematic after proof is finished.

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Higher Order: schematic variables can be functions.

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Higher Order Pattern:

- is a term in β normal form where
- each occurrence of a schematic variable is of the form $?f\ t_1\ \ldots\ t_n$
- and the $t_1\ \ldots\ t_n$ are η-convertible into n distinct bound variables

We have learned so far...

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➜ Typing rules for $\lambda^{\rightarrow}$, type variables, type contexts

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