DATA 61

COMP4161: Advanced Topics in Software Verification

HOL

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Last time...

- natural deduction rules for $\wedge$, $\vee$, $\longrightarrow$, $\neg$, iff...
- proof by assumption, by intro rule, elim rule
- safe and unsafe rules

- indent your proofs! (one space per subgoal)
- prefer implicit backtracking (chaining) or *rule_tac*, instead of *back*
- *prefer* and *defer*
- *oops* and *sorry*

Content

- ➜ Intro & motivation, getting started [1]

- ➜ Foundations & Principles
  - Lambda Calculus, natural deduction [1,2]
  - Higher Order Logic [3[a]]
  - Term rewriting [4]

- ➜ Proof & Specification Techniques
  - Inductively defined sets, rule induction [5]
  - Datatypes, recursion, induction [6, 7]
  - Hoare logic, proofs about programs, C verification [8[b],9]
  - (mid-semester break)
  - Writing Automated Proof Methods [10]
  - Isar, codegen, typeclasses, locales [11[c],12]

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[a] a1 due; [b] a2 due; [c] a3 due

Quantifiers

Scope

- Scope of parameters: whole subgoal
- Scope of $\forall, \exists, \ldots$: ends with ; or $\Longrightarrow$

Example:

$$\bigwedge x\ y.\ [\![\ \forall y.\ P\ y \longrightarrow Q\ z\ y;\quad Q\ x\ y\ ]\!] \implies \exists x.\ Q\ x\ y$$

means

$$\bigwedge x\ y.\ [\![\ (\forall y_1.\ P\ y_1 \longrightarrow Q\ z\ y_1);\quad Q\ x\ y\ ]\!] \implies (\exists x_1.\ Q\ x_1\ y)$$

Natural deduction for quantifiers

$$\frac{\bigwedge x.\ P\ x}{\forall x.\ P\ x}\ \text{allI} \qquad \frac{\forall x.\ P\ x \quad P\ ?x \Longrightarrow R}{R}\ \text{allE}$$

$$\frac{P\ ?x}{\exists x.\ P\ x}\ \text{exI} \qquad \frac{\exists x.\ P\ x \quad \bigwedge x.\ P\ x \Longrightarrow R}{R}\ \text{exE}$$

- **allI** and **exE** introduce new parameters ($\bigwedge x$).
- **allE** and **exI** introduce new unknowns ($?x$).

Instantiating Rules

apply (rule_tac x = "*term*" in *rule*)

Like **rule**, but ?*x* in *rule* is instantiated by *term* before application.

Similar: **erule_tac**

! *x* is in *rule*, not in goal !

Two Successful Proofs

1. $\forall x.\ \exists y.\ x = y$

apply (rule allI)

1. $\bigwedge x.\ \exists y.\ x = y$

best practice	exploration
apply (rule_tac x = "x" in exI)	**apply** (rule exI)
1. $\bigwedge x.\ x = x$	1. $\bigwedge x.\ x = ?y\ x$
apply (rule refl)	**apply** (rule refl)
	$?y \mapsto \lambda u.u$
simpler & clearer	**shorter & trickier**

Two Unsuccessful Proofs

$$1.\ \exists y.\ \forall x.\ x = y$$

apply (rule_tac x = ??? in exI)

apply (rule exI)

$1.\ \forall x.\ x = ?y$

apply (rule allI)

$1.\ \bigwedge x.\ x = ?y$

apply (rule refl)

$?y \mapsto x$ yields $\bigwedge x'.\ x' = x$

Principle:

$?f\ x_1 \ldots x_n$ **can only be replaced by term** t

if $params(t) \subseteq x_1, \ldots, x_n$

Safe and Unsafe Rules

Safe allI, exE

Unsafe allE, exI

Create parameters first, unknowns later

Demo: Quantifier Proofs

Parameter names

Parameter names are chosen by Isabelle

1. $\forall\, x.\ \exists y.\ x = y$

apply (rule allI)

1. $\bigwedge x.\ \exists y.\ x = y$

apply (rule_tac x = "x" in exI)

Brittle!

Renaming parameters

1. $\forall x.\ \exists y.\ x = y$

apply (rule allI)

1. $\bigwedge x.\ \exists y.\ x = y$

apply (rename_tac N)

1. $\bigwedge N.\ \exists y.\ N = y$

apply (rule_tac x = "N" in exI)

In general:
(rename_tac $x_1 \dots x_n$) renames the rightmost (inner) n parameters to $x_1 \dots x_n$

Forward Proof: frule and drule

apply (frule $<$ *rule* $>$)

Rule:	$[\![A_1; \ldots ; A_m]\!] \Longrightarrow A$
Subgoal:	1. $[\![B_1; \ldots ; B_n]\!] \Longrightarrow C$
Substitution:	$\sigma(B_i) \equiv \sigma(A_1)$
New subgoals:	1. $\sigma([\![B_1; \ldots ; B_n]\!] \Longrightarrow A_2)$
	$\vdots$
	m-1. $\sigma([\![B_1; \ldots ; B_n]\!] \Longrightarrow A_m)$
	m. $\sigma([\![B_1; \ldots ; B_n; A]\!] \Longrightarrow C)$

Like **frule** but also deletes B_i: **apply** (drule $<$ *rule* $>$)

Examples for Forward Rules

$$\frac{P \wedge Q}{P}\ \text{conjunct1} \qquad \frac{P \wedge Q}{Q}\ \text{conjunct2}$$

$$\frac{P \longrightarrow Q \quad P}{Q}\ \text{mp}$$

$$\frac{\forall x.\ P\ x}{P\ ?x}\ \text{spec}$$

Forward Proof: OF

$$r \ [\textbf{OF} \ r_1 \ldots r_n]$$

Prove assumption 1 *of theorem* r *with theorem* r_1*, and assumption* 2 *with theorem* r_2*, and* ...

Rule r	$[\![A_1; \ldots; A_m]\!] \Longrightarrow A$
Rule r_1	$[\![B_1; \ldots; B_n]\!] \Longrightarrow B$
Substitution	$\sigma(B) \equiv \sigma(A_1)$
r [OF r_1]	$\sigma([\![B_1; \ldots; B_n; A_2; \ldots; A_m]\!] \Longrightarrow A)$

Forward proofs: THEN

$$r_1 \text{ [THEN } r_2\text{]} \quad \text{means} \quad r_2 \text{ [OF } r_1\text{]}$$

Demo: Forward Proofs

Hilbert's Epsilon Operator

(David Hilbert, 1862-1943)

ε x. Px is a value that satisfies P (if such a value exists)

ε also known as **description operator**.
In Isabelle the ε-operator is written SOME x. P x

$$\frac{P\ ?x}{P\ (\text{SOME}\ x.\ P\ x)}\ \text{someI}$$

More Epsilon

$\mathcal{E}$ implies Axiom of Choice:

$$\forall x.\ \exists y.\ Q\ x\ y \Longrightarrow \exists f.\ \forall x.\ Q\ x\ (f\ x)$$

Existential and universal quantification can be defined with ε.

Isabelle also knows the definite description operator **THE** (aka ι):

$$\overline{(\text{THE } x.\ x = a) = a}\ \text{the\_eq\_trivial}$$

Some Automation

More Proof Methods:

apply (intro <intro-rules>)	repeatedly applies intro rules
apply (elim <elim-rules>)	repeatedly applies elim rules
apply clarify	applies all safe rules that do not split the goal
apply safe	applies all safe rules
apply blast	an automatic tableaux prover (works well on predicate logic)
apply fast	another automatic search tactic

Epsilon and Automation Demo

We have learned so far...

- Proof rules for predicate calculus
- Safe and unsafe rules
- Forward Proof
- The Epsilon Operator
- Some automation