COMP4161: Advanced Topics in Software Verification

$\longrightarrow$

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Content

- Intro & motivation, getting started [1]
- Foundations & Principles
  - Lambda Calculus, natural deduction [1,2]
  - Higher Order Logic [3[a]]
  - Term rewriting [4]
- Proof & Specification Techniques
  - Inductively defined sets, rule induction [5]
  - Datatypes, recursion, induction [6, 7]
  - Hoare logic, proofs about programs, C verification [8[b],9]
  - (mid-semester break)
  - Writing Automated Proof Methods [10]
  - Isar, codegen, typeclasses, locales [11[c],12]

[a] a1 due; [b] a2 due; [c] a3 due

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Last Time on HOL

➜ Defining HOL

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Last Time on HOL

- ➜ Defining HOL
- ➜ Higher Order Abstract Syntax

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- ➜ Deriving proof rules

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Last Time on HOL

- ➜ Defining HOL
- ➜ Higher Order Abstract Syntax
- ➜ Deriving proof rules
- ➜ More automation

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Term Rewriting

The Problem

Given a set of equations

$$
\begin{aligned}
l_1 &= r_1 \\
l_2 &= r_2 \\
&\vdots \\
l_n &= r_n
\end{aligned}
$$

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The Problem

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does equation $l = r$ hold?

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does equation $l = r$ hold?

Applications in:

- ➜ **Mathematics** (algebra, group theory, etc)
- ➜ **Functional Programming** (model of execution)
- ➜ **Theorem Proving** (dealing with equations, simplifying statements)

Term Rewriting: The Idea

use equations as reduction rules

$$l_1 \longrightarrow r_1$$
$$l_2 \longrightarrow r_2$$
$$\vdots$$
$$l_n \longrightarrow r_n$$

decide $l = r$ **by deciding** $l \stackrel{*}{\longleftrightarrow} r$

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Arrow Cheat Sheet

$$\xrightarrow{0} \quad = \quad \{(x, y) | x = y\} \qquad \text{identity}$$

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Arrow Cheat Sheet

$\xrightarrow{0}$	$=$	$\{(x,y) \mid x = y\}$	identity
$\xrightarrow{n+1}$	$=$	$\xrightarrow{n} \circ \longrightarrow$	n+1 fold composition

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$\longleftarrow$	$=$	$\xrightarrow{-1}$	inverse

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$\xleftrightarrow{+}$	$=$	$\bigcup_{i>0} \xleftrightarrow{i}$	transitive symmetric closure
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How to Decide $l \stackrel{*}{\longleftrightarrow} r$

Same idea as for β:

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Same idea as for β: look for n such that $l \stackrel{*}{\longrightarrow} n$ and $r \stackrel{*}{\longrightarrow} n$

Does this always work?

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If $l \stackrel{*}{\longrightarrow} n$ and $r \stackrel{*}{\longrightarrow} n$ then $l \stackrel{*}{\longleftrightarrow} r$. Ok.

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Rules: $f\ x \longrightarrow a, \quad g\ x \longrightarrow b, \quad f\ (g\ x) \longrightarrow b$

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But: $f\ x \longrightarrow a$ and $g\ x \longrightarrow b$ and a, b in normal form

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Works only for systems with **Church-Rosser** property:

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Fact: $\longrightarrow$ is Church-Rosser iff it is confluent.

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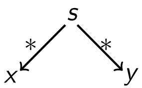
Confluence



Problem:
is a given set of reduction rules confluent?

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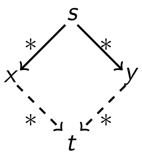

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undecidable

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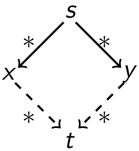

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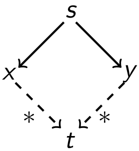
Local Confluence



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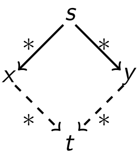

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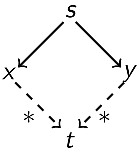

Fact: local confluence and termination $\Longrightarrow$ confluence

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Termination

$\longrightarrow$ is **terminating** if there are no infinite reduction chains
$\longrightarrow$ is **normalizing** if each element has a normal form
$\longrightarrow$ is **convergent** if it is terminating and confluent

Example:

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$\longrightarrow_\beta$ in λ is not terminating, but confluent

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Problem: is a given set of reduction rules terminating?

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When is $\longrightarrow$ Terminating?

Basic idea:

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When is $\longrightarrow$ Terminating?

Basic idea: when each rule application makes terms simpler in some way.

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More formally: $\longrightarrow$ is terminating when there is a well founded order $<$ on terms for which $s < t$ whenever $t \longrightarrow s$ (well founded = no infinite decreasing chains $a_1 > a_2 > \ldots$)

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② $<_r$ is well founded, because $<$ is well founded on $\mathbb{N}$

Termination in Practice

In practice: often easier to consider just the rewrite rules by themselves, rather than their application to an arbitrary term t.

Show

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$g\ x < f\ (g\ x)$ and $f\ x < g\ (f\ x)$

Requires

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u to become smaller whenever any subterm of u is made smaller.

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Requires $<$ to be **monotonic** with respect to the structure of terms:

$$s < t \longrightarrow u[s] < u[t].$$

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True for most orders that don't treat certain parts of terms as special cases.

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Example Termination Proof

Problem: Rewrite formulae containing $\neg$, $\wedge$, $\vee$ and $\longrightarrow$, so that they don't contain any implications and $\neg$ is applied only to variables and constants.

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Rewrite Rules:

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Rewrite Rules:

➜ Remove implications:

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➜ Remove implications:

imp: $(A \longrightarrow B) = (\neg A \vee B)$

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Rewrite Rules:

- Remove implications:

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- Push $\neg$s down past other operators:

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  notnot: $(\neg\neg P) = P$

  notand: $(\neg(A \wedge B)) = (\neg A \vee \neg B)$

  notor: $(\neg(A \vee B)) = (\neg A \wedge \neg B)$

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We show that the rewrite system defined by these rules is terminating.

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Order on Terms

Each time one of our rules is applied, either:

➜ an implication is removed, or

➜ something that is not a $\neg$ is hoisted upwards in the term.

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This suggests a 2-part order, $<_r$: $s <_r t$ iff:

- num_imps $s <$ num_imps t, or
- num_imps $s =$ num_imps $t \wedge$ osize $s <$ osize t.

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Let:

- $s <_i t \equiv$ num_imps $s <$ num_imps t and
- $s <_n t \equiv$ osize $s <$ osize t

Then $<_i$ and $<_n$ are both well-founded orders (since both return nats).

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This suggests a 2-part order, $<_r$: $s <_r t$ iff:

- num_imps $s <$ num_imps t, or
- num_imps $s =$ num_imps $t \wedge$ osize $s <$ osize t.

Let:

- $s <_i t \equiv$ num_imps $s <$ num_imps t and
- $s <_n t \equiv$ osize $s <$ osize t

Then $<_i$ and $<_n$ are both well-founded orders (since both return nats). $<_r$ is the lexicographic order over $<_i$ and $<_n$. $<_r$ is well-founded since $<_i$ and $<_n$ are both well-founded.

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Order Decreasing

imp clearly decreases num_imps.

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Order Decreasing

imp clearly decreases num_imps.

osize adds up all non-$\neg$ operators and variables/constants, weights each one according to its depth within the term.

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$$
\begin{array}{lll}
\text{osize}'\ c & x & = 2^x \\
\text{osize}'\ (\neg P) & x & = \text{osize}'\ P\ (x+1) \\
\text{osize}'\ (P \wedge Q) & x & = 2^x + (\text{osize}'\ P\ (x+1)) + (\text{osize}'\ Q\ (x+1)) \\
\text{osize}'\ (P \vee Q) & x & = 2^x + (\text{osize}'\ P\ (x+1)) + (\text{osize}'\ Q\ (x+1)) \\
\text{osize}'\ (P \longrightarrow Q) & x & = 2^x + (\text{osize}'\ P\ (x+1)) + (\text{osize}'\ Q\ (x+1)) \\
\text{osize}\ \ P & & = \text{osize}'\ P\ 0
\end{array}
$$

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\text{osize}\ \ P & & = \text{osize}'\ P\ 0
\end{array}
$$

The other rules decrease the depth of the things osize counts, so decrease osize.

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Term Rewriting in Isabelle

Term rewriting engine in Isabelle is called **Simplifier**

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Term Rewriting in Isabelle

Term rewriting engine in Isabelle is called **Simplifier**

apply simp

➜ uses simplification rules

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Term Rewriting in Isabelle

Term rewriting engine in Isabelle is called **Simplifier**

apply simp

- ➜ uses simplification rules
- ➜ (almost) blindly from left to right

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Term Rewriting in Isabelle

Term rewriting engine in Isabelle is called **Simplifier**

apply simp

- ➜ uses simplification rules
- ➜ (almost) blindly from left to right
- ➜ until no rule is applicable.

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Term Rewriting in Isabelle

Term rewriting engine in Isabelle is called **Simplifier**

apply simp

- ➜ uses simplification rules
- ➜ (almost) blindly from left to right
- ➜ until no rule is applicable.

termination: not guaranteed
(may loop)

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Term Rewriting in Isabelle

Term rewriting engine in Isabelle is called **Simplifier**

apply simp

- ➜ uses simplification rules
- ➜ (almost) blindly from left to right
- ➜ until no rule is applicable.

termination:	not guaranteed (may loop)
confluence:	not guaranteed (result may depend on which rule is used first)

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Control

- ➜ Equations turned into simplification rules with **[simp]** attribute

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Control

- Equations turned into simplification rules with **[simp]** attribute
- Adding/deleting equations locally:

  apply (simp add: <rules>)   and   **apply** (simp del: <rules>)

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Control

- ➜ Equations turned into simplification rules with **[simp]** attribute
- ➜ Adding/deleting equations locally:

  apply (simp add: <rules>) and **apply** (simp del: <rules>)
- ➜ Using only the specified set of equations:

  apply (simp only: <rules>)

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Demo

We have seen today...

➜ Equations and Term Rewriting

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We have seen today...

➜ Equations and Term Rewriting

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We have seen today...

➜ Equations and Term Rewriting

➜ Confluence and Termination of reduction systems

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We have seen today...

- ➜ Equations and Term Rewriting
- ➜ Confluence and Termination of reduction systems
- ➜ Term Rewriting in Isabelle

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Exercises

➜ Show, via a pen-and-paper proof, that the osize function is monotonic with respect to the structure of terms from that example.

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