



COMP4161: Advanced Topics in Software Verification

$\{P\} \dots \{Q\}$

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Content



- Intro & motivation, getting started [1]

- Foundations & Principles
 - Lambda Calculus, natural deduction [1,2]
 - Higher Order Logic [3^a]
 - Term rewriting [4]

- Proof & Specification Techniques
 - Inductively defined sets, rule induction [5]
 - Datatypes, recursion, induction [6, 7]
 - Hoare logic, proofs about programs, C verification [8^b,9]
 - (mid-semester break)
 - Writing Automated Proof Methods [10]
 - Isar, codegen, typeclasses, locales [11^c,12]

^aa1 due; ^ba2 due; ^ca3 due



DATA
61



A Crash Course in Semantics

**(For more, see the book
Concrete Semantics by
Tobias Nipkow and Gerwin
Klein)**

IMP - a small Imperative Language



Commands:
datatype com

=	SKIP	
	Assign vname aexp	{ - := - }
	Semi com com	{ - ; - }
	Cond bexp com com	{ IF - THEN - ELSE - }
	While bexp com	{ WHILE - DO - OD }

IMP - a small Imperative Language



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type_synonym vname	=	string	
type_synonym state	=	vname \Rightarrow nat	

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type_synonym vname	=	string	
type_synonym state	=	vname \Rightarrow nat	
type_synonym aexp	=	state \Rightarrow nat	
type_synonym bexp	=	state \Rightarrow bool	

Example Program



Usual syntax:

```
B := 1;  
WHILE A ≠ 0 DO  
    B := B * A;  
    A := A - 1  
OD
```


Example Program



Usual syntax:

```
 $B := 1;$   
WHILE  $A \neq 0$  DO  
   $B := B * A;$   
   $A := A - 1$   
OD
```

Expressions are functions from state to bool or nat:

```
 $B := (\lambda\sigma. 1);$   
WHILE  $(\lambda\sigma. \sigma A \neq 0)$  DO  
   $B := (\lambda\sigma. \sigma B * \sigma A);$   
   $A := (\lambda\sigma. \sigma A - 1)$   
OD
```

What does it do?

So far we have defined:



What does it do?



So far we have defined:

- **Syntax** of commands and expressions

What does it do?



So far we have defined:

- **Syntax** of commands and expressions
- **State** of programs (function from variables to values)

Now we need:

What does it do?



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- **State** of programs (function from variables to values)

Now we need: the meaning (semantics) of programs

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How to define execution of a program?

What does it do?



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- **State** of programs (function from variables to values)

Now we need: the meaning (semantics) of programs

How to define execution of a program?

- A wide field of its own

What does it do?



So far we have defined:

- **Syntax** of commands and expressions
- **State** of programs (function from variables to values)

Now we need: the meaning (semantics) of programs

How to define execution of a program?

- A wide field of its own
- Some choices:
 - Operational (inductive relations, big step, small step)
 - Denotational (programs as functions on states, state transformers)
 - Axiomatic (pre-/post conditions, Hoare logic)

Structural Operational Semantics



$$\overline{\langle \text{SKIP}, \sigma \rangle} \rightarrow \sigma$$

Structural Operational Semantics



$$\overline{\langle \text{SKIP}, \sigma \rangle \rightarrow \sigma}$$

$$\overline{\langle x := e, \sigma \rangle \rightarrow}$$

Structural Operational Semantics



$$\overline{\langle \text{SKIP}, \sigma \rangle \rightarrow \sigma}$$

$$\frac{e \sigma = v}{\langle x := e, \sigma \rangle \rightarrow \sigma[x \mapsto v]}$$

Structural Operational Semantics



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$$\overline{\langle c_1; c_2, \sigma \rangle \rightarrow \sigma''}$$

Structural Operational Semantics



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$$\frac{b \sigma = \text{True}}{\langle \text{IF } b \text{ THEN } c_1 \text{ ELSE } c_2, \sigma \rangle \rightarrow \sigma'}$$

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$$\frac{b \sigma = \text{False} \quad \langle c_2, \sigma \rangle \rightarrow \sigma'}{\langle \text{IF } b \text{ THEN } c_1 \text{ ELSE } c_2, \sigma \rangle \rightarrow \sigma'}$$

Structural Operational Semantics


$$\overline{\langle \text{WHILE } b \text{ DO } c \text{ OD}, \sigma \rangle \rightarrow}$$

Structural Operational Semantics



$$\frac{b \ \sigma = \text{False}}{\langle \text{WHILE } b \text{ DO } c \text{ OD}, \sigma \rangle \rightarrow \sigma}$$

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DATA
61



Demo: The Definitions in Isabelle

Proofs about Programs



Now we know:

- What programs are: Syntax
- On what they work: State
- How they work: Semantics

Proofs about Programs



Now we know:

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- On what they work: State
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So we can prove properties about programs

Proofs about Programs



Now we know:

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So we can prove properties about programs

Example:

Show that example program from slide 6 implements the factorial.

lemma $\langle \text{factorial}, \sigma \rangle \rightarrow \sigma' \implies \sigma' B = \text{fac } (\sigma A)$
(where $\text{fac } 0 = 1$, $\text{fac } (\text{Suc } n) = (\text{Suc } n) * \text{fac } n$)

Demo: Example Proof

Too tedious



Induction needed for each loop

Too tedious



Induction needed for each loop

Is there something easier?

Floyd/Hoare



Idea: describe meaning of program by pre/post conditions

Examples:

Floyd/Hoare



Idea: describe meaning of program by pre/post conditions

Examples:

$\{\text{True}\} \quad x := 2 \quad \{x = 2\}$

Floyd/Hoare



Idea: describe meaning of program by pre/post conditions

Examples:

$\{\text{True}\} \quad x := 2 \quad \{x = 2\}$

$\{y = 2\} \quad x := 21 * y \quad \{x = 42\}$

Floyd/Hoare



Idea: describe meaning of program by pre/post conditions

Examples:

$\{\text{True}\} \quad x := 2 \quad \{x = 2\}$

$\{y = 2\} \quad x := 21 * y \quad \{x = 42\}$

$\{x = n\} \quad \text{IF } y < 0 \text{ THEN } x := x + y \text{ ELSE } x := x - y \quad \{x = n - |y|\}$

Floyd/Hoare



Idea: describe meaning of program by pre/post conditions

Examples:

$\{\text{True}\} \quad x := 2 \quad \{x = 2\}$

$\{y = 2\} \quad x := 21 * y \quad \{x = 42\}$

$\{x = n\} \quad \text{IF } y < 0 \text{ THEN } x := x + y \text{ ELSE } x := x - y \quad \{x = n - |y|\}$

$\{A = n\} \quad \text{factorial} \quad \{B = \text{fac } n\}$

Floyd/Hoare



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$\{\text{True}\} \quad x := 2 \quad \{x = 2\}$

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$\{A = n\} \quad \text{factorial} \quad \{B = \text{fac } n\}$

Proofs: have rules that directly work on such triples

Meaning of a Hoare-Triple



$$\{P\} \quad c \quad \{Q\}$$

What are the assertions P and Q ?

Meaning of a Hoare-Triple



$$\{P\} \ c \ \{Q\}$$

What are the assertions P and Q ?

- Here: again functions from state to bool
(shallow embedding of assertions)

Meaning of a Hoare-Triple



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- Here: again functions from state to bool
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- Other choice: syntax and semantics for assertions (deep embedding)

What does $\{P\} \ c \ \{Q\}$ mean?

Meaning of a Hoare-Triple



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Partial Correctness:

$$\models \{P\} \ c \ \{Q\} \quad \equiv \quad \forall \sigma \ \sigma'. \ P \ \sigma \wedge \langle c, \sigma \rangle \rightarrow \sigma' \longrightarrow Q \ \sigma'$$

Meaning of a Hoare-Triple



$$\{P\} \ c \ \{Q\}$$

What are the assertions P and Q ?

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Partial Correctness:

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Total Correctness:

$$\models \{P\} \ c \ \{Q\} \quad \equiv \quad (\forall \sigma \ \sigma'. \ P \ \sigma \wedge \langle c, \sigma \rangle \rightarrow \sigma' \longrightarrow Q \ \sigma') \wedge \\ (\forall \sigma. \ P \ \sigma \longrightarrow \exists \sigma'. \ \langle c, \sigma \rangle \rightarrow \sigma')$$

Meaning of a Hoare-Triple



$$\{P\} \ c \ \{Q\}$$

What are the assertions P and Q ?

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Total Correctness:

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This lecture: partial correctness only (easier)

Hoare Rules


$$\overline{\{P\} \text{ SKIP } \{P\}}$$

Hoare Rules



$$\frac{}{\{P\} \text{ SKIP } \{P\}}$$

$$\frac{}{\{P[x \mapsto e]\} \ x := e \ \{P\}}$$

Hoare Rules



$$\frac{}{\{P\} \text{ SKIP } \{P\}} \quad \frac{}{\{P[x \mapsto e]\} \ x := e \ \{P\}}$$

$$\frac{\{P\} \ c_1 \ \{R\} \quad \{R\} \ c_2 \ \{Q\}}{\{P\} \ c_1; c_2 \ \{Q\}}$$

Hoare Rules



$$\frac{}{\{P\} \text{ SKIP } \{P\}} \quad \frac{}{\{P[x \mapsto e]\} \ x := e \ \{P\}}$$

$$\frac{\{P\} \ c_1 \ \{R\} \quad \{R\} \ c_2 \ \{Q\}}{\{P\} \ c_1; c_2 \ \{Q\}}$$

$$\frac{}{\{P\} \ \text{IF } b \ \text{THEN } c_1 \ \text{ELSE } c_2 \ \{Q\}}$$

Hoare Rules



$$\frac{}{\{P\} \text{ SKIP } \{P\}} \quad \frac{}{\{P[x \mapsto e]\} \text{ } x := e \text{ } \{P\}}$$

$$\frac{\{P\} c_1 \{R\} \quad \{R\} c_2 \{Q\}}{\{P\} c_1; c_2 \{Q\}}$$

$$\frac{\{P \wedge b\} c_1 \{Q\}}{\{P\} \text{ IF } b \text{ THEN } c_1 \text{ ELSE } c_2 \{Q\}}$$

Hoare Rules



$$\frac{}{\{P\} \text{ SKIP } \{P\}} \quad \frac{}{\{P[x \mapsto e]\} x := e \{P\}}$$

$$\frac{\{P\} c_1 \{R\} \quad \{R\} c_2 \{Q\}}{\{P\} c_1; c_2 \{Q\}}$$

$$\frac{\{P \wedge b\} c_1 \{Q\} \quad \{P \wedge \neg b\} c_2 \{Q\}}{\{P\} \text{ IF } b \text{ THEN } c_1 \text{ ELSE } c_2 \{Q\}}$$

Hoare Rules



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$$\frac{\{P\} c_1 \{R\} \quad \{R\} c_2 \{Q\}}{\{P\} c_1; c_2 \{Q\}}$$

$$\frac{\{P \wedge b\} c_1 \{Q\} \quad \{P \wedge \neg b\} c_2 \{Q\}}{\{P\} \text{ IF } b \text{ THEN } c_1 \text{ ELSE } c_2 \{Q\}}$$

$$\frac{\{P \wedge b\} c \{P\} \quad P \wedge \neg b \implies Q}{\{P\} \text{ WHILE } b \text{ DO } c \text{ OD } \{Q\}}$$

Hoare Rules



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$$\frac{\{P\} c_1 \{R\} \quad \{R\} c_2 \{Q\}}{\{P\} c_1; c_2 \{Q\}}$$

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$$\frac{\{P'\} c \{Q'\}}{\{P\} c \{Q\}}$$

Hoare Rules



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$$\frac{\{P\} c_1 \{R\} \quad \{R\} c_2 \{Q\}}{\{P\} c_1; c_2 \{Q\}}$$

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$$\frac{\{P \wedge b\} c \{P\} \quad P \wedge \neg b \implies Q}{\{P\} \text{ WHILE } b \text{ DO } c \text{ OD } \{Q\}}$$

$$\frac{P \implies P' \quad \{P'\} c \{Q'\} \quad Q' \implies Q}{\{P\} c \{Q\}}$$

Hoare Rules



$$\frac{}{\vdash \{P\} \text{ SKIP } \{P\}} \quad \frac{}{\vdash \{\lambda\sigma. P (\sigma(x := e \sigma))\} \quad x := e \quad \{P\}}$$

$$\frac{\vdash \{P\} \quad c_1 \quad \{R\} \quad \vdash \{R\} \quad c_2 \quad \{Q\}}{\vdash \{P\} \quad c_1; c_2 \quad \{Q\}}$$

$$\frac{\vdash \{\lambda\sigma. P \sigma \wedge b \sigma\} \quad c_1 \quad \{Q\} \quad \vdash \{\lambda\sigma. P \sigma \wedge \neg b \sigma\} \quad c_2 \quad \{Q\}}{\vdash \{P\} \quad \text{IF } b \text{ THEN } c_1 \text{ ELSE } c_2 \quad \{Q\}}$$

$$\frac{\vdash \{\lambda\sigma. P \sigma \wedge b \sigma\} \quad c \quad \{P\} \quad \bigwedge \sigma. P \sigma \wedge \neg b \sigma \implies Q \sigma}{\vdash \{P\} \quad \text{WHILE } b \text{ DO } c \text{ OD } \quad \{Q\}}$$

$$\frac{\bigwedge \sigma. P \sigma \implies P' \sigma \quad \vdash \{P'\} \quad c \quad \{Q'\} \quad \bigwedge \sigma. Q' \sigma \implies Q \sigma}{\vdash \{P\} \quad c \quad \{Q\}}$$

Are the Rules Correct?



Soundness: $\vdash \{P\} c \{Q\} \implies \models \{P\} c \{Q\}$

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Proof: by rule induction on $\vdash \{P\} c \{Q\}$

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Demo: Hoare Logic in Isabelle